

2.4. Separation of Variables - Spherical System

Steady-State heat conduction equation:

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 T}{\partial \phi^2} + \frac{1}{k} g(r, \theta, \phi) = 0$$

Another form: define new independent variable $\xi \equiv \cos \theta$

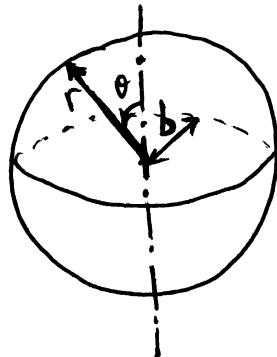
$$\begin{cases} \frac{df}{d\theta} = \frac{df}{d\xi} \cdot \frac{d\xi}{d\theta} = -\sin \theta \frac{df}{d\xi} \\ \frac{d^2f}{d\theta^2} = \frac{d}{d\theta} \left(-\sin \theta \frac{df}{d\xi} \right) = -\cos \theta \frac{df}{d\xi} + \sin^2 \theta \frac{d^2f}{d\xi^2} \end{cases}$$

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \xi} \left[(1-\xi^2) \frac{\partial T}{\partial \xi} \right] + \frac{1}{r^2 (1-\xi^2)} \frac{\partial^2 T}{\partial \phi^2} + \frac{1}{k} g(r, \xi, \phi) = 0$$

$$T = T(r, \xi, \phi)$$

* Example 1.

If the temperature distribution $T = f(\theta)$ is maintained over the entire surface of a sphere of radius b , what is the steady-state temperature at any point in the sphere? (no heat generation)



2D problem: $T(r, \theta)$ or $T(r, \xi)$
(not a function of ϕ)

The Complete problem:

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \xi} \left[(1-\xi^2) \frac{\partial T}{\partial \xi} \right] = 0 \quad \text{or, } \frac{\partial^2 T}{\partial r^2} + \frac{2}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial \xi} \left[(1-\xi^2) \frac{\partial T}{\partial \xi} \right] = 0$$

B.C. $\begin{cases} T|_{r=0} = \text{finite} \\ T|_{r=b} = f(\theta) \end{cases}$ i.e. $\begin{cases} T|_{r=0} = \text{finite} \\ T|_{r=b} = f(\theta) \text{ with } \xi = \cos \theta \end{cases}$

$\begin{cases} T|_{\theta=0} = \text{finite} \\ T|_{\theta=180^\circ} = \text{finite} \end{cases}$ and: $T|_{\xi=\pm 1} = \text{finite}$ Natural B.C.

① Separation of $T(r, \xi)$

Assume: $T(r, \xi) = R(r) \Theta(\xi)$

then: $R'' \Theta + \frac{2}{r} R' \Theta + \frac{1}{r^2} R \frac{d}{d\xi} \left[(1-\xi^2) \frac{d\Theta}{d\xi} \right] = 0$

$$\frac{r^2 R'' + 2r R'}{R} + \frac{d}{d\xi} \left[(1-\xi^2) \frac{d\Theta}{d\xi} \right] = 0 \quad (\Theta)$$

so: $\frac{r^2 R'' + 2r R'}{R} = - \frac{d}{d\xi} \left[(1-\xi^2) \frac{d\Theta}{d\xi} \right] = \mu \quad (\text{const.})$

therefore:

$$\begin{cases} r^2 R'' + 2r R' - \mu R = 0 \\ \frac{d}{d\xi} \left[(1-\xi^2) \frac{d\Theta}{d\xi} \right] + \mu \Theta = 0 \end{cases}$$

with homogeneous B.C.: $\begin{cases} R|_{r=0} = \text{finite} \\ \Theta|_{\xi=\pm 1} = \text{finite} \end{cases}$ (Natural B.C.)

(2) Solving ODEs.

First look at the equation for $\Theta(\xi)$ with $-1 \leq \xi \leq 1$.

$$\frac{d}{d\xi} \left[(1-\xi^2) \frac{d\Theta}{d\xi} \right] + \mu \Theta = 0$$

The natural boundary conditions at $\xi = \pm 1$ ($\theta = 0^\circ$ and 180°) require μ can only take certain values in order for the equation to have meaningful solutions.

i.e.: imposing the natural B.C. yields eigenvalues μ_n .

There are two important properties for the eigenvalues.

(1) $\mu_n \geq 0$

Proof: Let μ_n be the eigenvalue and $\Theta_n(\xi)$ the eigenfunction,

$$\frac{d}{d\xi} \left[(1-\xi^2) \frac{d\Theta_n}{d\xi} \right] + \mu_n \Theta_n = 0$$

$$\text{i.e., } \mu_n \Theta_n = - \frac{d}{d\xi} \left[(1-\xi^2) \frac{d\Theta_n}{d\xi} \right]$$

Multiplying each side with Θ_n and integrating over $\xi = \pm 1$:

$$\int_{-1}^{+1} \mu_n \Theta_n \cdot \underline{\Theta_n d\xi} = \int_{-1}^{+1} - \frac{d}{d\xi} \left[(1-\xi^2) \frac{d\Theta_n}{d\xi} \right] \cdot \underline{\Theta_n d\xi}$$

$$\begin{aligned} \mu_n \int_{-1}^{+1} \Theta_n^2 d\xi &= \int_{-1}^{+1} - \Theta_n d \left[(1-\xi^2) \frac{d\Theta_n}{d\xi} \right] \\ &= - \underbrace{\Theta_n (1-\xi^2) \Theta_n'}_{=0} \Big|_{\xi=-1}^{\xi=+1} + \int_{-1}^{+1} (1-\xi^2) [\Theta_n']^2 d\xi \end{aligned}$$

i.e.:

$$\begin{aligned} \mu_n \int_{-1}^{+1} \Theta_n^2 d\xi &= \int_{-1}^{+1} (1-\xi^2) \left(\frac{d\Theta_n}{d\xi} \right)^2 d\xi \\ &\geq 0 \quad \geq 0 \quad \geq 0 \end{aligned}$$

so: $\mu_n \geq 0$

$$(2) \quad \mu_n = n(n+1) \quad \text{with } n=0, 1, 2, \dots$$

If we let $\mu = n(n+1)$, the equation for $\Theta(\xi)$ becomes

$$\frac{d}{d\xi} \left[(1-\xi^2) \frac{d\Theta}{d\xi} \right] + n(n+1)\Theta = 0$$

i.e.:
$$(1-\xi^2)\Theta'' - 2\xi\Theta' + n(n+1)\Theta = 0 \quad \leftarrow \text{Legendre equation}$$

The solution has the form:

$$\begin{aligned} \Theta(\xi) &= Q_0 \left[1 - \frac{n(n+1)}{2!} \xi^2 + \frac{n(n-2)(n+1)(n+3)}{4!} \xi^4 - \dots \right] \\ &\quad + Q_1 \left[\xi - \frac{(n-1)(n+2)}{3!} \xi^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!} \xi^5 - \dots \right] \end{aligned}$$

where Q_0 and Q_1 are arbitrary constants.

Unless n is an integer that makes the series solution terminate with finite terms, the solution will diverge at either of the boundaries $\xi = \pm 1$.

Therefore: $n = 0, 1, 2, \dots$

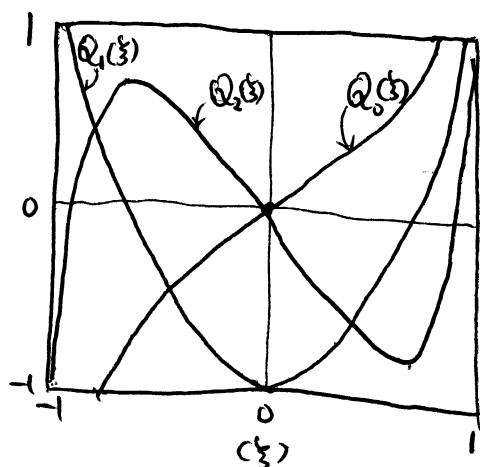
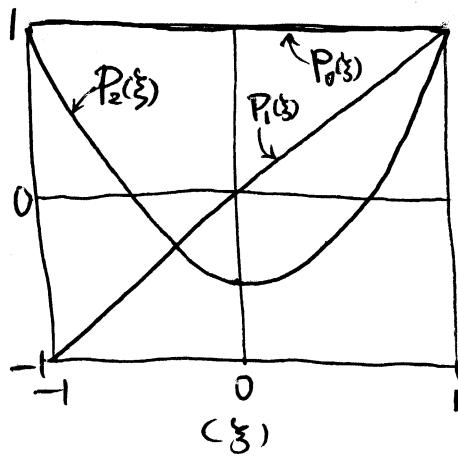
and $\mu_n = n(n+1)$

The equation for $\Theta(\xi)$ becomes:

$$(1-\xi^2)\Theta'' - 2\xi\Theta' + n(n+1)\Theta = 0 \quad \text{with } n=0, 1, 2, \dots$$

general solution: $\Theta(\xi) = C P_n(\xi) + D Q_n(\xi)$

$P_n(\xi)$ and $Q_n(\xi)$ are Legendre functions.



The equation of $R(r)$ becomes,

$$r^2 R'' + 2r R' - n(n+1)R = 0 \quad \text{with } n=0, 1, 2, \dots$$

general solution: $R(r) = Ar^n + \frac{B}{r^{n+1}}$

Imposing B.C. $\left. \mathcal{H} \right|_{\xi=\pm 1} = \text{finite}$ (again!)

because $\left. Q_n(\xi) \right|_{\xi=\pm 1}$ diverges $\Rightarrow B=0$, $\left. \mathcal{H}_n(\xi) = C P_n(\xi) \right.$

Imposing B.C. $\left. R \right|_{r=0} = \text{finite}$

because $\left. \frac{1}{r^{n+1}} \right|_{r=0}$ diverges $\Rightarrow B=0$, $R(r) = Ar^n$

Therefore, for each $n=0, 1, 2, \dots$

$$\overline{T}_n(r, \xi) = C_n r^n P_n(\xi)$$

i.e., $\overline{T}_n(r, \theta) = C_n r^n P_n(\cos \theta)$ (with C_n constant)

③ Making final solution

$$T(r, \theta) = \underbrace{\sum_{n=0}^{\infty} C_n r^n P_n(\cos\theta)}$$

④ Determining unknown coefficient.

Apply Nonhomogeneous B.C.: $T|_{r=b} = f(\theta)$

$$\text{therefore: } T|_{r=b} = \sum_{n=0}^{\infty} C_n b^n P_n(\cos\theta) = f(\theta)$$

Note: the orthogonal property of $P_n(\cos\theta)$:

$$\boxed{\int_0^\pi P_n(\cos\theta) P_m(\cos\theta) \sin\theta d\theta = \begin{cases} 0 & (m \neq n) \\ \frac{2}{2n+1} & (m = n) \end{cases}}$$

Using $\int_0^\pi P_m(\cos\theta) \sin\theta d\theta$ to operate both side of

$$f(\theta) = \sum_{n=0}^{\infty} C_n b^n P_n(\cos\theta)$$

$$\int_0^\pi f(\theta) P_m(\cos\theta) \sin\theta d\theta = \sum_{n=0}^{\infty} \int_0^\pi C_n b^n P_n(\cos\theta) P_m(\cos\theta) \sin\theta d\theta$$

$$\begin{aligned} \int_0^\pi f(\theta) P_n(\cos\theta) \sin\theta d\theta &= C_n b^n \int_0^\pi P_n^2(\cos\theta) \sin\theta d\theta \\ &= C_n b^n \cdot \left(\frac{2}{2n+1} \right) \end{aligned}$$

so:

$$C_n = \frac{2n+1}{2b^n} \int_0^\pi f(\theta) P_n(\cos\theta) \sin\theta d\theta$$

and:

$$T(r, \theta) = \sum_{n=0}^{\infty} \frac{2n+1}{2} \cdot \left(\frac{r}{b} \right)^n \cdot \int_0^\pi f(\theta') P_n(\cos\theta') \sin\theta' d\theta' \cdot P_n(\cos\theta)$$